Automorphisms of simple quotients of the Poisson and universal enveloping algebras of sl_2

Altyngul Naurazbekova

L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan

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This is a joint work with professor U. Umirbaev.

Let $\mathfrak{M}$ be some variety of linear algebras over a field K.

Let $F_n = K\langle x_1, x_2, \ldots, x_n\rangle$ be the free algebra of the variety $\mathfrak{M}$ with a free set of generators $\{x_1, x_2, \ldots, x_n\}$.

$\mathrm{Aut}\, F_n$ is the automorphism group of this algebra.

The automorphism φ of algebra F_n such that $\varphi(x_i) = f_i$, where $1 \leq i \leq n$, will be denoted by

$$\varphi = (f_1, f_2, \ldots, f_n).$$

Automorphisms of the form

$$\sigma(i, \alpha, f) = (x_1, x_2, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq \alpha \in K$, $f \in K\langle x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$, are called *elementary*.

The subgroup $T(F_n)$ of the group $\mathrm{Aut}\, F_n$ generated by all elementary automorphisms is called the *subgroup of tame automorphisms*.

Non-tame automorphisms are called *wild*.

H.W.E. Jung (1942): all automorphisms of the polynomial algebra $K[x, y]$ in two variables x, y over a field K of characteristic zero are tame.

W. van der Kulk (1953): all automorphisms of the polynomial algebra $K[x, y]$ in two variables x, y over a field K of arbitrary characteristic are tame.

Let G be an arbitrary group, G_1, G_2 be subgroups of the group G and $G_0 = G_1 \cap G_2$. The group G is called *the free product of the subgroups* G_1 *and* G_2 *with the amalgamated subgroup* G_0 and is denoted by $G = G_1 *_{G_0} G_2$ if

(a) G is generated by the subgroups G_1 and G_2;

(b) the defining relations of the group G consist only of the defining relations of the subgroups G_1 and G_2.

W. Van der Kulk (1953), I.R. Shafarevich (1966): the automorphism group $\mathrm{Aut}\, K[x,y]$ of algebra $K[x,y]$ admits an amalgamated free product structure, i.e.,

$$\mathrm{Aut}\, K[x,y] = A *_C B,$$

where A is the affine automorphism subgroup, B is the triangular automorphism subgroup, and $C = A \cap B$.

L. Makar-Limanov (1970), A.J. Czerniakiewicz (1971): all automorphisms of the free associative algebra $K\langle x, y\rangle$ in two variables x, y over field K are tame.

L. Makar-Limanov, U. Turusbekova, U. Umirbaev, (2009): all automorphisms of the free Poisson algebra $P\langle x, y\rangle$ in two variables x, y over the field K of characteristics zero are tame.

Moreover,

$$\mathrm{Aut}\, K[x, y] \cong \mathrm{Aut}\, K\langle x, y\rangle \cong \mathrm{Aut}\, P\langle x, y\rangle\,.$$

D. Kozybaev, L. Makar-Limanov, U. Umirbaev (2008): the automorphism group of the free right-symmetric algebras of rank two is tame.

A. Alimbaev, A. Naurazbekova, D. Kozybaev (2019): the automorphism group of the free right-symmetric algebras of rank two admits an amalgamated free product structure.

I. Shestakov, U. Umirbaev (2004): The well-known Nagata automorphism

$$\sigma = (x + 2y(zx - y^2) + z(zx - y^2)^2, y + z(zx - y^2), z)$$

of the polynomial algebra $K[x, y, z]$ over a field K of characteristic 0 is non-tame.

U. Umirbaev (2007): free associative algebras in three variables over a field of characteristic 0 also have non-tame automorphisms.

P. Cohn (1964): all automorphisms of finitely generated free Lie algebras over a field are tame.

J. Lewin (1968): this result was extended to free algebras of Nielsen-Schreier varieties.

Recall that a variety of algebras is called Nielsen-Schreier, if any subalgebra of a free algebra of this variety is free.

The varieties of all non-associative algebras (A.G. Kurosh), commutative and anti-commutative algebras (A.I. Shirshov), Lie algebras (A.I. Shirshov, E. Witt), and Lie superalgebras (A.A. Mikhalev, A.S. Stern) over a field are Nielsen-Schreier.

A. Alimbaev, A. Naurazbekova, D. Kozybaev (2019): the automorphism groups of free non-associative algebras and free commutative algebras of rank two admit an amalgamated free product structure.

A. Alimbaev, R. Nauryzbaev, U. Umirbaev (2020): The groups of automorphisms of free Lie algebras and free anti-commutative algebras of rank three also admit an amalgamated free product structure.

The study of relations between the automorphism groups of Poisson algebras and their deformation-quantizations is motivated by the *Belov-Kanel – Kontsevich Conjecture* (2005): the automorphism group of the Weyl algebra A_n of index $n \geq 1$ over a field K is isomorphic to the group of automorphisms of the symplectic Poisson algebra P_n, i.e.,

$$\mathrm{Aut}\, A_n \simeq \mathrm{Aut}\, P_n.$$

The Weyl algebra A_n of index $n \geq 1$, is the associative algebra over a field K with generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and defining relations

$$[Y_i, X_j] = \delta_{ij}, \quad [X_i, X_j] = 0, \quad [Y_i, Y_j] = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$.

The symplectic Poisson algebra P_n of index $n \geq 1$ is the polynomial algebra in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ endowed with the Poisson bracket defined by

$$\{y_i, x_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where $1 \leq i, j \leq n$.

J. Alev (1986), J. Dixmier (1968), L. Makar-Limanov (1984): The structure of the automorphism group of the Weyl algebra A_1 is described.

These results easily imply that the groups of automorphisms of the symplectic Poisson algebra P_1 and the Weyl algebra A_1 are isomorphic, i.e., the Belov-Kanel – Kontsevich Conjecture is true for $n = 1$.

A. Belov-Kanel, M. Kontsevich (2005): the groups of tame automorphisms of the symplectic Poisson algebra P_n and the Weyl algebra A_n are isomorphic.

A vector space P over a field K endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called a *Poisson algebra* if P is a commutative associative algebra under $x \cdot y$, P is a Lie algebra under $\{x, y\}$, and P satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = y \cdot \{x, z\} + \{x, y\} \cdot z.$$

Let L be an arbitrary Lie algebra with Lie bracket $[\,,]$ over a field K and let $e_1, e_2 \ldots$ be a linear basis of L. Then there exists a unique bracket $\{\,,\}$ on the polynomial algebra $K[e_1, e_2, \ldots]$ defined by $\{e_i, e_j\} = [e_i, e_j]$ for all i, j and satisfying the Leibniz identity. With this bracket

$$P(L) = \langle K[e_1, e_2, \ldots], \cdot, \{\,,\}\rangle$$

becomes a Poisson algebra. This Poisson algebra $P(L)$ is called the *Poisson enveloping algebra* (or the symmetric Poisson algebra) of L.

Let L be an arbitrary Lie algebra over a field K of characteristic zero. Denote by $U(L)$ the universal enveloping algebra of L and by $P(L)$ the Poisson enveloping algebra of L.

J. Dixmier (1977): the well known symmetrization map

$$S : P(L) \to U(L)$$

is an isomorphism of L-modules. Moreover, $U(L)$ is the most well-known natural deformation-quantization of $P(L)$.

Consider the three dimensional simple Lie algebra $\mathrm{sl}_2(K)$ over an algebraically closed field K of characteristic zero.

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is the standard basis of the Lie algebra $\mathrm{sl}_2(K)$. We have

$$[E, F] = H, [H, E] = 2E, [H, F] = -2F.$$

In 1973 J. Dixmier studied the quotients

$$U_\lambda = U(\mathrm{sl}_2(\mathbb{C}))/(C_U - \lambda)$$

of the universal enveloping algebra $U(\mathrm{sl}_2(\mathbb{C}))$ of the three dimensional simple Lie algebra $\mathrm{sl}_2(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$, where C_U is the standard Casimir element and $0 \neq \lambda \in \mathbb{C}$.

The structure of these algebras depend on λ and U_λ is simple if $\lambda \neq n^2 + 2n$ for any natural n. If $\lambda = n^2 + 2n$, then U_λ has a unique non-trivial ideal of finite codimension $(n+1)^2$.

In the same paper Dixmier described generators of the group of automorphisms of the algebra U_λ and defined the group of tame automorphisms of $U(\mathrm{sl}_2(\mathbb{C}))$. He also formulated a question on the existence of wild automorphisms of $U(\mathrm{sl}_2(\mathbb{C}))$.

In 1976 A. Joseph gave an example of a wild automorphism of $U(\mathrm{sl}_2(\mathbb{C}))$.

In 1998 O. Fleury represented the automorphism group of the quotient algebra U_λ as an amalgamated product of its subgroups.

Let $P(\mathrm{sl}_2(K))$ be the Poisson enveloping algebra of the Lie algebra $\mathrm{sl}_2(K)$ over an algebraically closed field K of characteristic zero.

The center of the Poisson enveloping algebra $P(\mathrm{sl}_2(K))$ of the Lie algebra $\mathrm{sl}_2(K)$ is equal to $K[C_P]$, where

$$C_P = 4EF + H^2$$

is the Casimir element of $\mathrm{sl}_2(K)$ in $P(\mathrm{sl}_2(K))$. This is an easy corollary of the fact that the center of the universal enveloping algebra $U(\mathrm{sl}_2(K))$ of $\mathrm{sl}_2(K)$ is generated by the standard Casimir element

$$C_U = 4FE + H^2 + 2H = 4EF + H^2 - 2H = 2EF + 2FE + H^2$$

of the universal enveloping algebra $U(\mathrm{sl}_2(K))$ of $\mathrm{sl}_2(K)$ and the image of C_P under the symmetrization map is C_U

U. Umirbaev, V. Zhelyabin (2021): the quotient algebras

$$P_\lambda = P(\mathrm{sl}_2(K))/(C_P - \lambda),$$

$0 \neq \lambda \in K$, are simple.

Denote by $\varphi = (f_1, f_2, f_3)$, where $f_1, f_2, f_3 \in K[E, H, F] = P(\mathrm{sl}_2(K))$, the automorphism of the Poisson algebra $P(\mathrm{sl}_2(K))$ such that $\varphi(E) = f_1, \varphi(H) = f_2, \varphi(F) = f_3$.

If $\theta = (f_1, f_2, f_3)$ and $\varphi = (g_1, g_2, g_3)$, then the product in $\mathrm{Aut}(P(\mathrm{sl}_2(K)))$ is defined by

$$\theta \circ \varphi = (g_1(f_1, f_2, f_3), g_2(f_1, f_2, f_3), g_3(f_1, f_2, f_3)).$$

Let $A = \mathrm{Aut}(\mathrm{sl}_2(K))$ be the group of all automorphisms of the Lie algebra $\mathrm{sl}_2(K)$. It is clear that any automorphism of the algebra $\mathrm{sl}_2(K)$ can be uniquely extended to an automorphism of the Poisson algebra $P(\mathrm{sl}_2(K))$.

It is well known that every automorphism of the algebra $\mathrm{sl}_2(K)$ is inner, i.e., every automorphism coincides with

$$\widehat{T} : \mathrm{sl}_2(K) \to \mathrm{sl}_2(K), X \mapsto T^{-1}XT$$

for some matrix $T \in \mathrm{GL}_2(K)$.

Consequently, $A \simeq PSL_2(K) = GL_2(K)/\{\alpha I | \alpha \in K^*\} = SL_2(K)/\{I, -I\}$, where I is the identity matrix.

A linear automorphism $\widehat{T}$ is called *triangular* if T is a lower triangular matrix, i.e.,

$$T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in GL_2(K).$$

Denote by C the subgroup of all triangular automorphisms of $\mathrm{sl}_2(K)$.

Lemma 1

$C \simeq C_1 \rtimes \mathrm{Hr}$, where

$$C_1 = \{\Delta_\alpha = (E - \alpha H - \alpha^2 F, H + 2\alpha F, F) | \alpha \in K\}$$

and

$$\mathrm{Hr} = \{H_\beta = (\beta E, H, 1/\beta F) | \beta \in K^*\}.$$

The automorphisms H_β in Lemma 1 are called *hyperbolic rotations* and the group Hr will be called the group of *hyperbolic rotations*.

Lemma 2

The system of elements

$$A_0 = \{\tau_\alpha = (F, -H + 2\alpha F, E + \alpha H - \alpha^2 F) | \alpha \in K\} \cup \{\mathrm{id} = (E, H, F)\}$$

is a system of representatives of the left cosets of C in A.

Lemma 3

The group $A = \mathrm{Aut}(\mathrm{sl}_2(K))$ is generated by

$$\tau(E) = F, \tau(H) = -H, \tau(F) = E$$

and the automorphisms of the form

$$\Delta_\alpha(E) = E - \alpha H - \alpha^2 F,$$

$$\Delta_\alpha(H) = H + 2\alpha F,$$

$$\Delta_\alpha(F) = F,$$

where $\alpha \in K$.

Lemma 4

Every automorphism of the algebra $\mathrm{sl}_2(K)$ fixes the element C_P of $P(\mathrm{sl}_2(K))$.

For any fixed $\lambda \in K$ denote by $(C_P - \lambda)$ the principal ideal of the polynomial algebra $P(\mathrm{sl}_2(K)) = K[E, H, F]$ generated by $C_P - \lambda$. Since C_P belongs to the center of the Poisson algebra $P(\mathrm{sl}_2(K))$, it follows that $(C_P - \lambda)$ is a Poisson ideal of $P(\mathrm{sl}_2(K))$.

Denote by $P_\lambda = P(\mathrm{sl}_2(K))/(C_P - \lambda)$ the quotient algebra of the Poisson algebra $P(\mathrm{sl}_2(K))$ by the ideal $(C_P - \lambda)$. Denote by e, h, f the images of E, H, F in P_λ, respectively. Then we have

$$\{e, f\} = h, \{h, e\} = 2e, \{h, f\} = -2f, 4ef + h^2 = \lambda. \tag{3.1}$$

Notice that P_λ, as an associative and commutative algebra, is generated by e, h, f and defined by one relation

$$ef = -\frac{1}{4}h^2 + \frac{1}{4}\lambda. \tag{3.2}$$

Let $G = \mathrm{Aut}(P_\lambda)$ be the group of all Poisson automorphisms of the Poisson algebra P_λ. By Lemma 4, every automorphism of $\mathrm{sl}_2(K)$ induces a unique automorphism of $\mathrm{Aut}(P_\lambda)$. Further we will identify $A = \mathrm{Aut}(\mathrm{sl}_2(K))$ with the corresponding subgroup of G.

L. Makar-Limanov (1990): description of generators of the automorphism group of the quotient algebra

$$R = K[x, y, z]/(xy - P(z))$$

of the polynomial algebra $K[x, y, z]$ over an algebraically closed field K, where $P(z) \in K[z]$.

Notice that P_λ as an associative and commutative algebra is represented as $R = K[f, e, h]/(fe - P(h))$, where $P(h) = -\frac{1}{4}h^2 + \frac{\lambda}{4}$.

Proposition 1

The group $G = \mathrm{Aut}(P_\lambda)$ is generated by

$$\tau(e) = f, \tau(h) = -h, \tau(f) = e$$

and the automorphisms of the form

$$\begin{aligned} \Delta_g(e) &= e - g(f)h - g^2(f)f, \\ \Delta_g(h) &= h + 2g(f)f, \\ \Delta_g(f) &= f, \end{aligned} \tag{3.3}$$

where $g(x) \in K[x]$.

Denote by J the subgroup of all automorphisms of the form (3.3) of the group $G = \mathrm{Aut}(P_\lambda)$. Denote by $T = \mathrm{Tr}(P_\lambda)$ the subgroup of $G = \mathrm{Aut}(P_\lambda)$ generated by the subgroup J and by the subgroup of hyperbolic rotations Hr. The subgroup T will be called the subgroup of *triangular* automorphisms of P_λ.

Lemma 5

$$T = J \rtimes \mathrm{Hr}.$$

Recall that C is the subgroup of linear triangular automorphisms of $\mathrm{Aut}(\mathrm{sl}_2(K))$.

Lemma 6

The system of elements

$$B_0 = \{\Delta_q \mid q(x) \in xK[x]\}$$

is a system of representatives of the left cosets of C in T.

Let G be an arbitrary group, G_1, G_2 be subgroups of the group G and $G_0 = G_1 \cap G_2$. If S_i is a system of left representatives of G_i by G_0 including the identity element 1, then the group G is a free product of subgroups G_1 and G_2 with the amalgamated subgroup G_0 if and only if every element of G can be uniquely represented in the form

$$g_1 \ldots g_k c,$$

where $g_i \in (S_1 \cup S_2) \setminus \{1\}$, $i = 1, \ldots, k$, g_i, g_{i+1} do not belong to S_1 and S_2 at the same time, and $c \in G_0$.

Lemma 7

Let A_0 and B_0 be the sets defined in Lemma 2 and Lemma 6, respectively. Then any automorphism ϕ of the algebra P_λ decomposes into a product of the form

$$\phi = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda, \tag{4.1}$$

where $\alpha_i \in A_0$, $\alpha_2, \ldots, \alpha_k \neq \mathrm{id}$, $\beta_i \in B_0$, $\beta_1, \ldots, \beta_k \neq \mathrm{id}$, $\lambda \in C$.

Lemma 8

The decomposition (4.1) of the automorphism ϕ from Lemma 7 is unique.

Theorem 1

The group G of automorphisms of the Poisson algebra P_λ is the free product of subgroups A and T with amalgamated subgroup $C = A \cap T$, i.e.,

$$G = A *_C T.$$

For any $\lambda \in K$ let $U_\lambda = U(\mathrm{sl}_2(K))/(C_U - \lambda)$ be the quotient algebra of the algebra $U(\mathrm{sl}_2(K))$ by the principal ideal $(C_U - \lambda)$. Denote by e, h, f the images of E, H, F in U_λ, respectively. Then we have

$$[e, f] = h, [f, h] = 2f, [h, e] = 2e, \lambda = 4fe + h^2 + 2h = 4ef + h^2 - 2h = 2ef + 2fe + h^2. \quad (5.1)$$

Theorem 2 J. Dixmier (1973)

The group $\mathrm{Aut}(U_\lambda)$ is generated by the exponential automorphisms $\Phi_{n,\mu}$ and $\Psi_{n,\mu}$ $(n > 0, \mu \in K)$, where

$$\Phi_{n,\mu} = (e - \mu n f^{n-1} h + \mu n(n-1) f^{n-1} - \mu^2 n^2 f^{2n-1}, h + 2\mu n f^n, f),$$

$$\Psi_{n,\mu} = (e, h - 2\mu n e^n, f + \mu n e^{n-1} h + \mu n(n-1) e^{n-1} - \mu^2 n^2 e^{2n-1}).$$

Recall that $A = \mathrm{Aut}(\mathrm{sl}_2(K))$ is the group of all automorphisms of the Lie algebra $\mathrm{sl}_2(K)$. In Section 3 we identified A with a subgroup of automorphisms $G = \mathrm{Aut}(P_\lambda)$ of the Poisson algebra P_λ.

J. Dixmier (1973): Every automorphism of $\mathrm{sl}_2(K)$ gives a unique automorphism of the algebra U_λ.

For this reason we identify A with the corresponding subgroup of the automorphism group of U_λ. Thus, without loss of generality, we can assume that $\mathrm{Aut}(P_\lambda)$ and $\mathrm{Aut}(U_\lambda)$ both contain A.

Denote by J' the subgroup of all automorphisms $\Phi_{n,\mu}$ $(n > 0, \mu \in K)$ of $\mathrm{Aut}(U_\lambda)$. Notice that every automorphism $\Phi_{n,\mu}$ can be written in the form

$$\begin{aligned} \delta_g(e) = e - g(f)h - fg^2(f) + fg'(f), \\ \delta_g(h) = h + 2fg(f), \\ \delta_g(f) = f, \end{aligned} \tag{5.2}$$

where $g(x) \in K[x]$ and $g'(x)$ is the formal derivative of $g(x)$.

Denote by $T' = \mathrm{Tr}(P_\lambda)$ the subgroup of $\mathrm{Aut}(U_\lambda)$ generated by the subgroup J' and by the subgroup of hyperbolic rotations Hr. The subgroup T' will be called the subgroup of *triangular* automorphisms of U_λ.

Theorem 3 O. Fleury (1998)

We have

$$\mathrm{Aut}(U_\lambda) = A *_C T',$$

where $A = \mathrm{Aut}(\mathrm{sl}_2(K))$ is the subgroup of all automorphisms of the Lie algebra $\mathrm{sl}_2(K)$ in $\mathrm{Aut}(U_\lambda)$ and $C = A \cap T'$.

Lemma 9

$$T' = J' \rtimes \mathrm{Hr}.$$

Theorem 4

The automorphism groups of the algebras P_λ and U_λ are isomorphic, i.e.,

$$\mathrm{Aut}(P_\lambda) \cong \mathrm{Aut}(U_\lambda).$$

Unfortunately, the question on the isomorphism of the automorphism groups of $P(\mathrm{sl}_2(K))$ and $U(\mathrm{sl}_2(K))$ remains open.

Thank you for your attention!